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MATHS403 (GENERAL TOPOLOGY 1)

COURSE OUTLINE

1 Topological spaces: Definition and examples of topological spaces, open and closed sets, neighbourhoods, limits (cluster) points, Interior and Closure of a set, boundary, Coarser and finer topologies, Bases and subbasis, Subspaces of topological spaces. product topology.

2 Quotient topology: First and second Countable spaces, separable spaces, separation axioms. Topology of metric spaces, Convergence of sequence in a topological space, pointwise and uniform Convergence, limit of functions at given points. Limit of function in first Countable Hausdorff spaces.

3 Continuous mapping: Continuity in metric spaces, open and closed mappings. Homeomorphism. Topological invariants.

4 Connectedness: Union, product, Closure of Connected sets Intervals as Connected subsets of the real line. Image of Connected sets under Continuous mapping, Connected Component

REFERENCE

i General topology, Kelly JL springer India (2005)

ii Topology (A first Course) James R. munkor, P.H Inc (1975)

iii General topology schaum's outline series Seymour Lipschitz

CHAPTER ONE

1 TOPOLOGICAL SPACES

The power set of a set X, denoted by $P(X)$ or 2^X is the set containing all possible subsets of X.

Example: Let $X = \{a, b, c\}$ then the power set

$P(X) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \emptyset\} = 2^X$

If X is finite i.e X has n-elements then $P(X)$ will have 2^n elements. In the example above $|P(X)| = 2^X = 2^3 = 8$

DEFINITION 1.1: Let X be a non-empty set. A Class τ of subsets of X is called topology on X iff τ satisfies the following axioms:

i X and $\emptyset$ belongs to τ i.e $X, \emptyset \in \tau$

ii The Union of any members of τ, is also in τ i.e $\bigcup_{S_i \in \tau} \in \tau$

iii The Intersection of any two elements of τ belongs to τ i.e $S_1 S_2 \in \tau \Rightarrow S_1 \cap S_2 \in \tau$

Now, the set X together with topology defined on it i.e the pair (X, τ) is called a topological space.

NB: A metric is a function while topology is a Collection of finite subset of a non-empty sets.

Examples: Usual topological space

1. The class $\mathcal{U}$ of all open sets of real numbers i.e. $\{(a,b)$ with $x \in \mathbb{R} \ni a < x < b\}$ is called a usual topology on $\mathbb{R}$ and the pair $(\mathbb{R}, \mathcal{U})$ is a topological space called the usual topological space.

2. The class $P(X)$ of all possible subsets of a set X is a topology called the discrete topology and therefore $(X, P(X)) = (X, 2^X) = (X, P)$ is called discrete topological space.

NB: i. This is the biggest topological space we can have in set

ii. The topology is the biggest topology we can have in the set

3. For any set X, the collection $\tau = \{\emptyset, X\}$ form a topology called Indiscrete topology.

NB: This topology is the smallest topology we can have in a set.

4. Let X be a set and τ be the class of all subsets of X whose complements are finite together with the empty set. Then τ is a topology called Co-finite topology.

Let $X = \{a, b, c\}$

$P(X) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \emptyset\}$

The subsets of X whose complement are finite together with the empty set (τ) are

$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$

Now, τ is called co-finite topology and (X, τ) is called the co-finite topological space.

NB: If the set X is finite, then the co-finite topology coincide with the discrete topology.

Exercise 1: i) Which of the following classes of subsets of X form topology. $X = \{a, b, c, d, e\}$

$\tau_1 = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$

$\tau_2 = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$

$\tau_3 = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{a,b,d,e\}\}$

ii) Confirm whether the following classes are indeed topology

$\tau_4 = \{\cup [a,b) : a, b \in \mathbb{R}, a < b\}$ i.e. lower limit topology LLT

$\tau_5 = \{\cup (a,b] : a, b \in \mathbb{R}, a < b\}$ i.e. upper limit topology ULT

$\tau_6 = \{$ Intersection of any two topologies on $X\}$

$\tau_7 =$ Intersection of any collection of topologies on X

OPEN SET IN TOPOLOGICAL SPACE

DEFINITION 1.2: Let (X, τ) be a topological space. A subset G of X $G(X)$ is said to be an open set in (X, τ) iff G is an element of τ i.e. $(G \in \tau)$.

CLOSED SET IN TOPOLOGICAL SPACE

DEFINITION 1.3: A subset F of X, $F(X)$ is said to be closed in (X, τ) iff F^c is a subset of τ i.e. either $F^c \subseteq \tau$ or $F^c \in \tau$.

Example: Suppose $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a,b\}\}$.

Then the open sets are $= \{\phi, X, \{a\}, \{a,b\}\}$

Then the closed sets are $= \{X, \phi, \{b,c\}, \{c\}\}$

Thus, that set ϕ and X are both open and closed.

But $\{a, c\}$ is neither open nor closed.

Exercise 2: Show that if F_1 and F_2 are closed set in topological space (X, τ), then $F_1 \cup F_2$ is also closed in topological space (X, τ).

LIMIT POINT

Definition 1.4: Let (X, τ) be a topological space and $A \subseteq X$. An element $x \in X$ is called a limit point of A, iff for every open set $G \in \tau \ni x \in G$, we have $A \cap \{G \setminus \{x\}\} \neq \phi$.

Example: Suppose $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$. What are the limit point of $A = \{a, b, c\}$

Soln A open set containing $a = X, \{a\}, \{a,c,d\}$

- $b = \{b,c,d,e\}, X$
- $c = \cancel{X}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}$
- $d = X, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}$
- $e = X, \{b,c,d,e\}$

Therefore, b, d, e are limit point of A.

OR

19/1/22

$X = \{a,b,c,d,e\}$ and $\tau = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$.

$A = \{a,b,c\} \Rightarrow a \in \{X, \{a\}, \{a,c,d\}\} \Rightarrow X \cap \{A - \{a\}\} \neq \phi$

$a \to X, \{a\}, \{a,c,d\} \Rightarrow A \cap \{X \setminus \{a\}\} \neq \phi$

$A \cap \{\{a\} \setminus \{a\}\} \neq \phi \Rightarrow a$ is not a limit point of A.

$b \to X, \{b,c,d,e\} \Rightarrow A \cap \{X \setminus \{b\}\} \neq \phi$

$A \cap \{\{b,c,d,e\} \setminus \{b\}\} \neq \phi \to b$ is not a limit point of A

$c \to X, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}$

$A \cap \{X \setminus \{c\}\} \neq \phi$

$A \cap \{\{a,c,d\} \setminus \{c\}\} \neq \phi$

$\Rightarrow c$ is not a limit point of A

$d \to X, \{c,d,b\}, \{a,c,d\}, \{b,c,d,e\}$

$A \cap \{X \setminus \{d\}\} = \phi$

$A \cap \{c,d\} \setminus \{d\} = \phi, \{\{a,c,d\} \setminus \{d\}\} \neq \phi$

$\therefore A \cap (\{b,c,d,e\} \setminus \{d\}) \neq \phi$

$\Rightarrow d$ is not a limit point of A.

$e \to X, \{b,c,d,e\}$

$A \cap (X \setminus \{e\}) \neq \phi$

$A \cap (\{b,c,d,e\} \setminus \{e\}) \neq \phi$

$\Rightarrow e$ is not a limit point of A.

DERIVED SET OF A

…tion 1.5: In any topological space (X, τ), the set of all limit points of set $A \subseteq X$ denoted as A', is called the Derived set of A.

…mple: Consider the discrete space (X, τ). If p is an element in X then $\{p\}$ is an open set for any $A \subseteq X$ and $p \in X$, we have $A \cap \{p \setminus p\} = \emptyset$

$\Rightarrow p$ cannot be a limit point of A. Since p is arbitrary then (X, τ) cannot have a limit point

$\Rightarrow A' = \emptyset$ for any $A \subseteq X$

On the other hand, if we consider Indiscrete space (X, I) then for any $A \subseteq X$, if $|A| \geq 2$, then suppose $a \in X$, we will have $X \cap \{A \setminus a\} \neq \emptyset$ which implies $A' = X$ since a is arbitrary.

But however, if $|A| = 1$, then suppose $a \in X$, we will have $X \cap \{A \setminus a\} = \emptyset$ if $a \in A$ and

$X \cap \{A \setminus a\} \neq \emptyset$ if $a \notin A$.

$\Rightarrow A' = X - A = A^c$.

…n 1.5: Let (X, τ) be a topological space. If $A \subseteq X$, then A is closed iff its contain all its limit point i.e A is closed iff $A' \subseteq A$

: Suppose A is closed, then A^c is open, we want to show that any element of A^c is not a limit point meaning that all limit points are in A.

This is clearly true, since for any $x \in A^c$ we have $A \cap \{A^c \setminus x\} = \emptyset$ because $A \cap A^c = \emptyset$.

If $A \cap B = \emptyset$ then $A \subseteq B^c$ or $B \subseteq A^c$

Thus, $A' \subseteq A$.

Conversely, suppose that $A' \subseteq A$, we need to show that A is closed. Then for $x \in A^c$, $x \notin A'$ from $A' \subseteq A$

$\Rightarrow \exists$ ~~at least~~ open ~~one~~ set say G_x $\ni$

$A \cap \{G_x \setminus x\} = \emptyset$

$\Rightarrow G_x \cap A = \emptyset$, since $x \notin A$.

$\Rightarrow x \in G_x \subseteq A^c$

$A^c = \{x \ni x \in A^c\} \subseteq \bigcup_{x \in A^c} G_x \subseteq A^c$

$\Rightarrow A^c = \bigcup_{x \in A^c} G_x$

$\Rightarrow A^c$ is open set

$\Rightarrow A$ is closed

Exercise 3: i) prove that if $A \subseteq B$, then $A' \subseteq B'$

ii) $(A \cup B)' = A' \cup B'$

iii) $A \cup A'$ is closed

In all case $A, B \in (X, \tau)$ a topological space.

CLOSURE OF A SET

Definition 1.6: Let (X, τ) be a topological space and $A \subseteq X$. The intersection of all closed set containing A, is called closure of A. Denoted by $\bar{A}$.

i.e $\bar{A} = \cap F_i$, F_i is closed and $F_i \supseteq A$.

REMARK i) $\bar{A}$ is closed, since it is intersection of closed sets.

ii) $\bar{A}$ is the small closed set containing A.

iii) $A \subseteq \bar{A}$

iv) If A is closed then $A = \bar{A}$

THEOREM 1.7: Let (X, τ) be a topological space and $A \subseteq X$. Then $\bar{A} = A \cup A'$.

Proof: $\bar{A}$ is the smallest closed set containing A. Also from (Exercise 3) $A \cup A'$ is closed and it contains A

$$\Rightarrow \bar{A} \subseteq A \cup A' \quad \text{--- (1)}$$

On the other hand $A \subseteq \bar{A} \Rightarrow A' \subseteq (\bar{A})'$ (by Exercise 3)

$A' \subseteq (\bar{A})' \subseteq \bar{A} \Rightarrow A' \subseteq \bar{A}$ and so

$$A \cup A' \subseteq A \quad \text{--- (2)}$$

From (1) and (2) we have $\bar{A} = A \cup A'$ $\square$

Examples: 25th/1/2022

i) For the discrete space (X, D), if $A \subseteq X$,

$A' = \emptyset$, $\bar{A} = A \cup A' = A \cup \emptyset = A$.

ii) For the indiscrete space (X, I), if $A \subseteq X$ and $|A| \geq 2$, $A' = X$

$\therefore \bar{A} = A \cup A' = A \cup X = X$

But if $|A| = 1$,

$A' = A^c$

$\therefore \bar{A} = A \cup A' = A \cup A^c = X$.

Defini

INTERIOR POINT AND INTERIOR OF A SET

DEFINITION 1.7: Let (X, τ) be a topological space and $A \subseteq X$, an element $x \in A$ is an interior point of A, if $\exists$ open $G_x \in \tau \ni x \in G_x \subseteq A$

The set of all interior points of A is called the interior of A denoted by A° or A or $\text{Int}(A)$

REMARK: i) A° is open

ii) $A^\circ \subseteq A$

iii) $A = A^\circ$ if A is open

THEOREM 1.8: Let (X, τ) be a topological space and $A \subseteq X$.

i) The interior of A, i.e A° of A is an open set contained in A

ii) A° is the largest open set contained in A

iii) A is open iff $A = A^\circ$

Proof: i) Suppose $x \in A^\circ \Rightarrow \exists\, G_x \in \tau \ni x \in G_x \subseteq A$.

On the other hand, if $y \in G_x$ then $y \in G_x \subseteq A \Rightarrow y \in A^\circ$

$\Rightarrow G_x \in A^\circ$. Consequently, $A^\circ = \bigcup_{x \in A} G_x \Rightarrow A^\circ$ is open.

ii) Suppose $G \in \tau$ and $G \subseteq A$. For any $x \in G$ we have $x \in G \subseteq A \Rightarrow x \in A^\circ \Rightarrow G \subseteq A^\circ$.

Thus, A° is the largest open set contained in A

Exercise 4 iii) **proof:** Since Int A is open, if $\text{Int} A = A$, then A is open.

Conversely, if A is open, then A is an element of G such that G is open and $G \subseteq A$.

So $\text{Int} A = A$.

EXTERIOR POINT AND BOUNDARY POINT

DEFINITION 1.8: i) Let (X,τ) be a topological space and $A \subseteq X$. An element $x \in X$ is said to be an exterior point of A if it is an interior point of A^c i.e $\exists\, G \in \tau \ni x \in G \subseteq A^c$.
The exterior of A is the set of all interior point of A^c. Thus, the exterior of A is the interior of A^c, written as $Ext(A) = Int(A^c) = (A^c)^\circ$.

ii) An element $x \in X$ which is neither in the interior of A nor in the exterior of A is called the boundary point. The set of all boundary point of A is called the boundary of A, denoted as $b(A)$.

Thus, $X = A^\circ \cup Ext(A) \cup b(A)$

* THEOREM 1.9: Let (X,τ) be a topological space and $A \subseteq X$ then (i) $(\bar{A})^c = (A^c)^\circ$ and (ii) $(A^\circ)^c = \overline{(A^c)}$

Proof: Let $x \in (\bar{A})^c \Rightarrow x \in (A \cup A')^c$
$\Rightarrow x \notin A$ and $x \notin A'$
$\Rightarrow \exists\, G \in \tau \ni x \in G$ and $A \cap (G - \{x\}) = \emptyset$
$\Rightarrow A \cap G = \emptyset$ since $x \notin A \Rightarrow G \subseteq A^c$ or $x \in G \subseteq A^c$.
$\Rightarrow x \in (A^c)^\circ \Rightarrow (\bar{A})^c = (A^c)^\circ$ ---- (i)

Conversely, if $x \in (A^c)^\circ \Rightarrow \exists\, G \in \tau \ni x \in G \subseteq A^c \Rightarrow G \cap A = \emptyset$
$\Rightarrow A \cap (G - \{x\}) = \emptyset$ since $x \notin A$
$\Rightarrow x \notin A'$ and $x \notin A$
$\Rightarrow x \notin A \cup A' \Rightarrow x \in (A \cup A')^c = x \in (\bar{A})^c$ --- (ii) $\square$

EXERCISE: Let (X,τ) be a topological space and $A \subseteq X$
i) Show that $(A^\circ)^c = \overline{(A^c)}$
ii) prove that $b(A) = \bar{A} - A^\circ$

DENSE SUBSET AND NOWHERE DENSE SUBSET

DEFINITION 1.9: Let (X,τ) be a topological space and $A \subseteq X$.
i A is said to be a dense subset of X, iff $\bar{A} = X$.
ii A is said to be nowhere dense in X, if $(\bar{A})^\circ = \emptyset$.

Example: On the real line $\mathbb{R}$, the set of rational number $\mathbb{Q}$ is dense in $\mathbb{R}$ i.e $\bar{\mathbb{Q}} = \mathbb{R}$. however $\mathbb{Q} \neq \mathbb{R}$.
On the other hand, if a subset say A is finite then $\bar{A} = A$. but $(\bar{A})^\circ = \emptyset$ i.e A is nowhere dense

NEIGHBOURHOOD

DEFINITION 1.10: Let (X,τ) be a topological space and an element $x \in X$. $N \subseteq X$ is said to be a neighbourhood of x iff $\exists\, G \in \tau \ni x \in G \subseteq N$.

Example: Consider $[a-\delta, a+\delta]$ where $a \in \mathbb{R}$ and $\delta > 0$ since $a \in (a-\delta, a+\delta) \subseteq [a-\delta, a+\delta]$. Then $[a-\delta, a+\delta]$ is a neighbourhood of a.

RELATIVE TOPOLOGY

DEFINITION 1.12: Let (X,τ) be a topological space and $A \subseteq X$. Let $\tau_A \subseteq 2^X$ be the set containing all intersection of A with open set i.e $B \in \tau_A$ iff $B = A \cap G$ for some $G \in \tau$. τ_A is a topology on A called Relative topology on A.

SUB-SPACE

The pair (A, τ_A) is then called a subspace of (X, τ).
To show that τ_A is a topology on A, we see that,

i) $\emptyset \in \tau_A$ since $\emptyset \cap A = \emptyset \Rightarrow \emptyset \in \tau_A$ and $\emptyset \in \tau$

ii) $A \in \tau_A$ since $A \cap X = A$, $X \in \tau$

iii) suppose $B_i \in \tau_A \Rightarrow B_i = G_i \cap A$ for some $G_i \in \tau$.
for each $i \ \exists \ B_i = G_i \cap A$
then $\bigcup_i B_i = \bigcup_i (G_i \cap A) = A \cap \bigcup_i G_i = A \cap G$ for some $G \in \tau$ since τ is a topology $\Rightarrow \bigcup_i B_i \in \tau_A$.

iv) Let B_1 and $B_2 \in \tau_A \Rightarrow \exists \ G_1$ and $G_2 \in \tau$
$\ni B_1 = A \cap G_1$ and $B_2 = A \cap G_2$. Then ~~for some~~
$B_1 \cap B_2 = (A \cap G_1) \cap (A \cap G_2)$
$= A \cap (G_1 \cap G_2) = A \cap G$ for some $G \in \tau$ since τ is a topology.
$\therefore B_1 \cap B_2 \in \tau_A$
$\Rightarrow \tau_A$ is a topology.

Example Let $X = \{a, b, c, d, e\}$ and $A = \{a, d, e\}$
If $\tau = \{X, \emptyset, \{a\}, \{d\}, \{a,c,d\}, \{b,c,d,e\}\}$
then $\tau_A = \{A, \emptyset, \{a\}, \{d\}, \{a,d\}, \{d,e\}\}$

DEF 1.13: FINNER AND COASER TOPOLOGY

DEF 1.13: Let τ_1 and τ_2 be two topologies on a non-empty set X. τ_2 is said to be finner than τ_1 (or τ_1 is coaser than τ_2) iff $\tau_1 \subseteq \tau_2$.
In the same way τ_1 is then said to be cosrser than τ_2.

NB If one topology is neither finner nor coarser than the other one, then the two topologies are said to be incomparable or not comparable.

Example Consider the discrete and the indiscrete topologies. It is clear that the indiscrete topology is a subset of the discrete topology $\Rightarrow$ the indiscrete topology is coaser than the discrete topology or the discrete topology is finner than the indiscrete topology. Certainly, the indiscrete topology is inside any other topology. Therefore, the indiscrete topology is coarser than any other topology.
On the other hand the discrete topology contains any other topology (on the same set).

Exercise: ~~If none of the two~~

i) Let $\{\tau_i, i \in I\}$ be any collection of topology on a set X, show that the intersection of the τ_i is also a topology on X.

ii) Let τ_1 and τ_2 be two topologies on X. Give an example to show that $\tau_1 \cup \tau_2$ is not a topology.

1/2/22

② CHAPTER TWO
BASES AND SUB-BASES

DEF 2.1 Let (X, τ) be a topological space. A subset $(\mathcal{B}, \tau)$ i.e. $\mathcal{B} \subseteq \tau$ is called a base for the topology if every member of τ is either in $\mathcal{B}$ or a union of member of $\mathcal{B}$.

Example: The set $\mathcal{B} = \{(a,b) / a,b \in \mathbb{R}\}$ is a base for the usual topology.

ii- The set $\mathcal{B}_2 = \{[a,b) / a,b \in \mathbb{R}\}$ is a base for lower limit topology (L.L.T).

iii- The set $\mathcal{B}_3 = \{(a,b] / a.b \in \mathbb{R}\}$ is a base for the upper limit topology (U.L.T)

iv- $\mathcal{B}_4 = \{(a,b) / a,b \in \mathbb{Q}\}$ is also a base for the usual topology.

v- $\mathcal{B}_5 = \{\{p\} / p \in X\}$ i.e the singleton sets, form a base for the discrete topology.

CORE 2.1.1 Let (X, τ) be a topological space and $\mathcal{B}$ a subset of 2^X i.e $\mathcal{B} \subseteq 2^X$. ~~Where~~ $X \neq \emptyset$.

EX 7 The collection $\mathcal{B}$ is a base for some topological on 2^X iff

i- $X = \bigcup_{b \in \mathcal{B}} b$

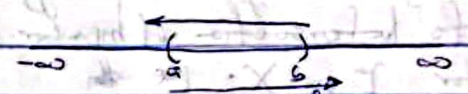
ii- If $b_1, b_2 \in \mathcal{B}$

$\Rightarrow b_1 \cap b_2$ is a union of member of $\mathcal{B}$.

DEF 2.2: SUBBASE

Let (X, τ) be a topological space. A class $\mathcal{S}$ of open subset of X is said to be a subbase for τ, iff finite intersection of its members form a base for τ.

Example: The intersection of (a, ∞) and $(-\infty, b)$ where $a < b$ is $(a, \infty) \cap (-\infty, b) = (a, b)$



This implies, since the class of open interval (a,b) form a base for the usual topology on $\mathbb{R}$ then the class of all infinite open intervals of the form (a, ∞) and $(-\infty, b)$ for a subbase for the usual topology on $\mathbb{R}$.

LOCAL BASE FOR τ AT x

DEF 2.3: Let (X, τ) be a topological space and $x \in X$. If L_x is a subset of τ such that for any open set $G \in \tau$ where $x \in G$, $\exists\ L_1 \in L_x \ni x \in L_1 \subset G$. Then L_x is called a local base at x or local base for τ at x.

Example: The set of all open balls containing the element in all open sets is a local base for the element.

EX 8 Let $(X, \mathcal{B})$ be a topological space and $\mathcal{B}$ be a base for τ. The collection $\mathcal{B}^* = \{B \cap A$ where $B \in \mathcal{B}\}$ is a base for the relative topology τ_A where $A \subset X$.

NB: Let A be any class of subsets of X i.e. $A \subseteq 2^X$. A may or may not be a base for X. However, A can be made to generate a topology on X as follows:

THEOREM 2.4: Any class A of subsets of $X \neq \emptyset$ is a subbase for a unique topology τ on X i.e. the finite intersection of member of A form a base for τ on X.

Example: Consider the following class of subsets of $X = \{a, b, c, d\}$

$A = \{\{a,b\}, \{b,c\}, \{d\}\}$. To do this we take the intersection of members of A i.e.

Intersection $I = \{\{a,b\}, \{b,c\}, \{d\}, \{b\}, \emptyset, X\}$

NB: ϕ and X are I because of the topological assertion that intersection of all singleton is $\emptyset$ and they are union is X. On the other hand union of all empty is empty and they intersection is always X.

The union of member of I is

$\tau = \{\{a,b\}, \{b,c\}, \{d\}, \{b\}, \emptyset, X, \{a,b,c\}, \{a,b,d\}, \{b,d\}, \{b,c,d\}\}$ is a topology generated by A.

THEOREM 2.5: Let (X,τ) be a topological space and A a class of subsets of X (i.e. $A \subseteq 2^X$). The τ^* generated by the finite intersections of members of A is exactly the intersection of all topologies on X which contain in A.

proof: Let $\{\tau_i\}$ be the collection of all topologies of X containing A and $\tau^* = \bigcap_{i \in k} \tau_i$. The topology τ^* contains A. Also τ^* contains A, implies

$\tau^* \subseteq \tau^*$ ---- (1)

On the other hand suppose $G \in \tau^*$, then by definition, $G = \bigcup_i (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_n})$ for some K and where $S_{i_k} \in A$ since $A \in \tau^*$ then $S_{i_k} \in \tau^*$

Since τ^* is a topology, then $S_{i_1} \cap S_{i_2} \cap \dots S_{i_k} \in \tau^*$ for the same reason $\bigcup_i (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_n}) \in \tau^*$

$\Rightarrow G \in \tau^*$

Consequently, $\tau^* \subseteq \tau^*$ ---- (2)

from (1) and (2), the result follows.

3 CHAPTER THREE
CONTINUITY IN TOPOLOGICAL SPACES

DEF 3.1: Let (X,τ) and (Y,τ^*) be two topological spaces and $f: (X,\tau) \to (Y,\tau^*)$ a function. f is said to be continuous at a point $x \in X$, if for any open set $H \subseteq Y$, where $f(x) \in H$, $\exists$ an open set G in X $\ni x \in G \subseteq f^{-1}(H)$.

: If $f:(X,\tau)\to(Y,\tau^*)$ is continuous at a point x, then $f_1:(X,\tau)\to(Y,\tau_1^*)$ will remain continuous provided $\tau_1^*\subseteq\tau^*$.

Similarly $f_2:(X,\tau_1)\to(Y,\tau^*)$ will remain continuous provided $\tau_1\supseteq\tau$. X

A function is said to be continuous on the whole of X, if it is continuous at each point of X.

Thm 3.2: Let $f:(X,\tau)\to(Y,\tau^*)$ be a function from a topological space (X,τ) into a (Y,τ^*). The function f is continuous in X, iff for each open set H of Y, $f^{-1}(H)$ is open in X.

proof:

Suppose for each $H\in Y$, $f^{-1}(H)$ is open in X. Let $x\in X$ and H be an open set in Y $\ni f(x)\in H$. From hypothesis $f^{-1}(H)$ as open set so let

$G=f^{-1}(H)\Rightarrow x\in f^{-1}(H)\subseteq f^{-1}(H)$

$\Rightarrow f(x)$ is continuous at x since x is arbitrary then the function f is continuous on X.

Conversely, if $f(x)$ is continuous on X and let H be any open set in Y; if $f^{-1}(H)=\phi$, then we are done since ϕ is open.

However, if $f^{-1}(H)\neq\phi$ and $x\in f^{-1}(H)$ then by continuity of f $\exists$ an open $G_x\ni x\in G_x\subseteq f^{-1}(H)$.

$\therefore f^{-1}(H)=\{x/x\in f^{-1}(H)\}\subseteq\bigcup G_x\subseteq f^{-1}(H)$

$\Rightarrow x\in f^{-1}(H)$

$\Rightarrow f^{-1}(H)=\bigcup_{x\in f^{-1}(H)} G_x$

$\Rightarrow f^{-1}(H)$ is open.

Examples 8/2/22

Open set is (Y,τ): $f^{-1}(X)=X$, $f^{-1}(\phi)=\phi$ which is open in X

1) Suppose $f:(X,\tau)\to(Y,\tau^*)$ and τ is discrete topology then f is continuous whatever τ^* is.

2) In similar way, if τ^* is the indiscrete topology then f is continuous whatever τ is.

3) Consider the restriction $f/A:(A,\tau_A)\to(Y,\tau^*)$ where $A\subseteq X$, if f is continuous on X, then f/A is also continuous on A.

DEF 3.3 Let $f:(X,\tau)\to(Y,\tau^*)$ be a function from topological space (X,τ) to the topological space (Y,τ^*), then

i. f is said to be open if every open set G on X, $f(G)$ is open set in Y;

ii. f is said to be closed if every closed set $F=X$, $f(F)$ is closed in Y.

EX9: Let $f:(X,\tau) \to (Y,\tau^*)$ be a function, show that f is continuous on X iff for every closed set F in Y, $f^{-1}(F)$ is also closed in X.

NB: for any two sets A and $B \subseteq X$ is a topological space (X,τ), then

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$$

HOMEOMORPHISM

DEF 3.4: A $f:(X,\tau) \to (Y,\tau^*)$ is called a homeomorphism if it is

i) one to one

ii) onto

iii) open

iv) continuous i.e it is bijective and b. continuous OR bijective where f and f^{-1} are continuous.

In that case, the topological space (X,τ) is then said to be homeomorphic to the topological space (Y,τ^*) or (X,τ) is equivalent to (Y,τ^*) denoted as $(X,\tau) \cong (Y,\tau^*)$.

Examples

i. Let $f(x) = (b-a)x+a$ where $f:[0,1] \to [a,b]$ both sets with usual topology.

Solution

$f(x) = (b-a)x+a$

Now, $f(0) = a$ $f(1) = b$, thus f is both one to one and onto. Moreover f^{-1} exist i.e $f^{-1}(y) = \frac{y-a}{b-a}$ and f^{-1} is also one to one & onto. Since $b>a$. f is continuous and f^{-1} is also continuous, which means f is open. Therefore f is one to one, onto, continuous and open and so it is a homeomorphism.

$\Rightarrow ([0,1], \mathcal{U}) \cong ([a,b], \mathcal{U})$.

Note ~~However that~~ $[0,1] \neq [a,b]$.

ii. Consider $f: X \to \mathbb{R}$ where $X = (-1,1)$ defined by $f(x) = \tan \frac{\pi x}{2}$.

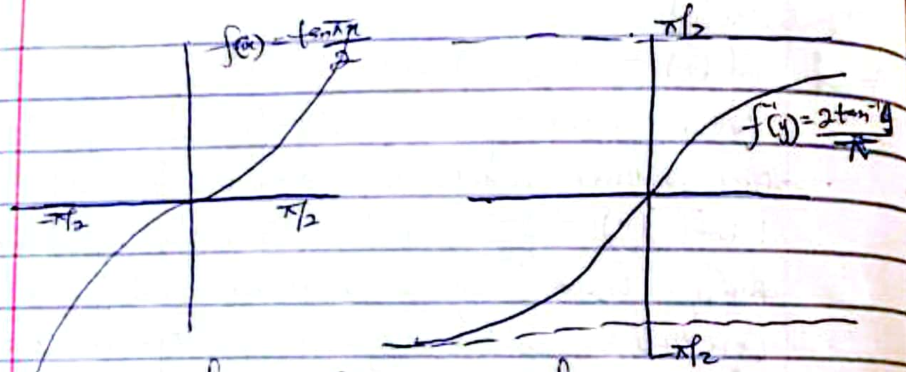
Solution

The function $f(x)$ is one to one and onto. It is also continuous. Moreover f^{-1} exists i.e $f^{-1}(y) = \frac{2\tan^{-1} y}{\pi}$. Also f^{-1} is continuous

$\Rightarrow f(x)$ is open.

i.e $f:[-1,1] \to \mathbb{R}$, $f(-1) = -\infty$, $f(1) = +\infty$

Therefore, $f(x)$ is one to one, onto ~~[illegible]~~ $f(x)$ is ~~clearly continuous~~ continuous and open.



EX10: prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous functions on X and Y respectively then $g \circ f : X \to Z$ is also continuous on X.

TOPOLOGICAL PROPERTY

DEF 3.5: Any property of the topological space which is preserved under homeomorphism is called a topological property. The property is then said to be topological invariant and every space which is homeomorphic to the topological space has that property.

Eg: Looking at our last example (i) we note that $[0,1] \cong [a,b]$ but the length of $[0,1]$ is not the same as the length as $[a,b]$ unless if $a=0$ or $b=1$.

In a similar way, we note that from example (ii) above noted that $(-1,1) \cong \mathbb{R}$ but the length $(-1,1)$ is different from $\mathbb{R}$. This shows that length is not a topological property. Similarly boundedness is not a topological property because from example (ii) $[-1,1]$ is bounded but $\mathbb{R}$ is not bounded.

4 CHAPTER FOUR

METRIC TOPOLOGY

DEF 4.1: Let X be a non-empty set. A real valued function d defined on $X \times X$ i.e. $d : X \times X \to \mathbb{R}$ is called a metric or distance function iff its satisfies the following axioms.

i) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$, $\forall x,y \in X$
ii) $d(x,y) = d(y,x)$ $\forall x,y \in X$
iii) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$

Examples

i) $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(a,b) = |a-b|$ is a metric called the usual metric on $\mathbb{R}$. ~~and the usual metric on $\mathbb{R}^2$~~

ii) The usual metric on $\mathbb{R}^2$ is given as
$d(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$

iii) on $\mathbb{R}^n$ $d(a,b) = \sqrt{(a_1-b_1)^2 + \cdots (a_n-b_n)^2}$

iv) Let X be any non-empty set and d be a function defined as $d(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$

Then d is a metric on X called the trivial metric.

v) Consider the class of continuous function $C[0,1]$ on the closed interval $[0,1]$. A metric can be defined on $C[0,1]$ as follows.

$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ which is precisely the area between the curves f and g from 0 to 1.

vi) Another metric on $[0,1]$ is as follows

$d^*(f,g) = \sup\{|f(x) - g(x)|\}$ which is precisely the ~~biggest~~ vertical gap between $f(x)$ and $g(x)$ from 0 to 1.

NB Any set X on which a metric d is defined is called a metric space and is denoted as (X, d).

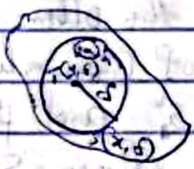
OPEN SPHERE

15/2/22

DEF 4.2: Let (X, d) be a metric space. If $x \in X$ and $\delta \in \mathbb{R}$ where $\delta > 0$, then the set $S(x,\delta) = \{y \in X \ni d(x,y) < \delta\}$ called an open sphere with center x and radius δ.

LEMMA 4.2.1

Let (X, d) be a metric space and $S(x,\delta)$ an open sphere with center x and radius δ. If $y \in S(x,\delta)$, then $\exists$ an open sphere $T(y,\delta_1)$ whose center is y and radius $\delta_1 \ni T(y,\delta_1) \subset S(x,\delta)$.



proof:

Since $y \in S(x,\delta)$ then $d(x,y) < \delta$. Hence $\delta - d(x,y) > 0$.
Choose $\delta_1 \ni 0 < \delta_1 < \delta - d(x,y)$. We claim that the open sphere $T(y,\delta_1) \subset S(x,\delta)$.

To prove ⊂, we see that if $z \in T(y,\delta_1)$, then $d(y,z) < \delta_1 \Rightarrow d(x,z) \leq d(x,y) + d(y,z)$

i.e. $d(x,z) < d(x,y) + \delta_1 < d(x,y) + \delta - d(x,y) = \delta$

$\Rightarrow d(x,z) < \delta \Rightarrow T(y,\delta_1) \subset S(x,\delta)$

$\Rightarrow z \in S(x,d)$.

LEMMA 4.2.2

Let S_1 and S_2 be two open spheres in a metric space (X,d). If $p \in S_1 \cap S_2$, then $\exists$ an open sphere $S_p \ni S_p \in S_1 \cap S_2$ with center p.

Proof:

Since $p \in S_1$, then from lemma 4.2.1, $\exists$ an open sphere S_1^* center $p \ni p \in S_1^* \subset S_1$.

Similarly $\exists$ an open sphere S_2^* center $p \ni p \in S_2^* \subset S_2$.

Since S_1^* and S_2^* have the same center, then one of them is inside the other (see the set of exercises). Without loss of generality (W.L.O.G). Let $S_1^* \subseteq S_2^*$

$\Rightarrow p \subset S_1^* \subset S_2^*$ and $p \subset S_1^* \subset S_2^* \subset S_2$

Lastly, if we choose $S_p = S_1^*$ then

$S_p \subset S_1$ and $S_p \subset S_2$

$\Rightarrow S_p \subset S_1 \cap S_2$.

THEOREM 4.2.3

Let $X \neq \emptyset$ and (X,d) a metric space. The class of open spheres $\beta = \{S(x,\delta) \ni x \in X, \delta \in \mathbb{R} \text{ and } \delta > 0\}$ in X forms a base for some topology on X.

Proof:

The proof follows from lemma 4.2.1, lemma 4.2.2 and theorem 2.1.1

REMARK

Theorem 4.2.3 show that every metric space has a topology associated with it, which is called the metric topology and it is the topology induced by the metric d.

Thus every metric space (X,d) is a topological space $(X,\mathcal{T}_d)$ where $\mathcal{T}_d$ is the topology induced by the metric d.

Accordingly therefore, all concepts defined for topological space such as open sets, closed sets, accumulation points, closure, interior and so on are also defined for metric space.

Examples

Consider the metric space (X,d) where

i) $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$, which is the trivial metric

The open spheres for these metrics are:

$S(x,\delta) = \begin{cases} \{x\} & \text{if } \delta \leq 1 \\ X & \text{if } \delta > 1 \end{cases}$

The induced topology here is the discrete topology.

ii) Considering the usual metric d on $\mathbb{R}$ i.e the metric space is $(\mathbb{R},d)$ where $d(x,y) = |x-y|$.

The open spheres are the usual open sets.

$S(x,\delta) = (x-\delta, x+\delta)$.

The topology induced here is the usual topology.

iii) Considering the usual metric on $\mathbb{R}^2$ given by

$\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ or $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$d((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

Exercise

Determine the open space

Two bases B_1 and B_2 are for the same topology iff every element of B_1 is a union of members of B_2 and vice-versa.

EQUIVALENT METRICS

DEF 4.3 Two metrics on a set X are said to be equivalent if they generate the same topology on X.

i.e $X \Rightarrow$ set $\quad d_1, d_2 \Rightarrow$ metrics.

Examples

Consider the following three metrics on $\mathbb{R}^2$.

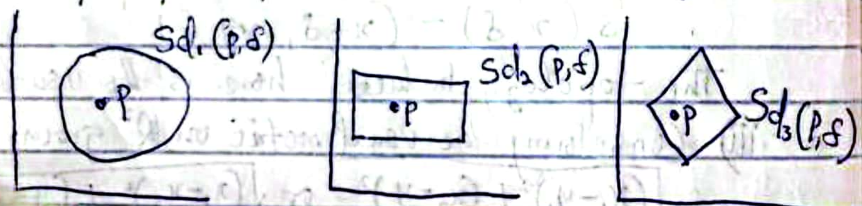
i) $d_1(p,q) = d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

ii) $d_2(p,q) = d((x_1,y_1),(x_2,y_2)) = \max(|x_1-x_2|, |y_1-y_2|)$

iii) $d_3(p,q) = d((x_1,y_1),(x_2,y_2)) = |x_1-x_2| + |y_1-y_2|$

The three metrics induce the same topology on $\mathbb{R}^2$ which is the usual topology on $\mathbb{R}^2$.

The open sphere of the three metrics are as follows:



Each of the three classes of open spaces is a [illegible] for the usual topology on $\mathbb{R}^2$.

DEF 4.4: Suppose (X,d) is a metric space and $x \in X$. If $A, B \subseteq X$ then we define ~~$d(x,A)=$~~

i $d(x,A) = \inf\{d(x,y) \ni y \in A\}$

Note that if $x \in A$, then $d(x,A) = 0$

ii Here $d(A,B) = \inf\{d(x,y) \ni x \in A, y \in B\}$

Note also that $A \cap B \neq 0$ then $d(A,B) = 0$

However, note that if $A \cap B = \emptyset$ it does not necessarily mean $d(A,B) \neq 0$

iii $d(A) = \sup\{d(x,y), x,y \in A\}$

Note if A has only one element then $d(A) = 0$

* THEOREM 4.4.1

Let (X,d) be a metric space and $x \in X$. Then $x \in \bar{A}$ iff $d(x,A) = 0$ i.e $\bar{A} = \{x \in X \ni d(x,A) = 0\}$.

proof

Suppose $d(x,A) = 0$, then any open sphere whose center is x, contains atleast one element of A. Consequently, any open set G containing x also contains at least one element of A. i.e $A \cap G \neq \emptyset$

$$\Rightarrow A \cap (G \setminus \{x\}) \neq \emptyset$$

$$\Rightarrow x \in A' \Rightarrow x \in \bar{A} = A \cup A'.$$

Conversely, if $d(x,A) = \varepsilon > 0$ then construct the open sphere $S(x, \varepsilon/2)$ one can say this open sphere contains no element of A or the ~~open~~ sphere is completely outside A.

i.e. $S(x, \varepsilon/2) \cap A = \emptyset$ or

$A \cap (S(x, \varepsilon/2) \setminus \{x\}) = \emptyset$

Since $x \notin A \Rightarrow x \notin A'$

$\Rightarrow x \notin A \cup A' = \bar{A}$

$\Rightarrow x \notin \bar{A}$ or $x \in \text{int}(A) \Rightarrow x \notin \bar{A}$.

COROLLARY 4.4.2

If A is a closed set and $x \notin A$, then $d(x, A) \neq 0$.

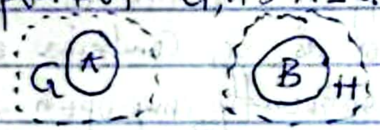
proof (exercise)

A - closed then $x \notin A'$ and $x \notin A$

$\Rightarrow d(x, A) \neq 0$

THEOREM 4.5 (A SEPARATION AXIOM).

Let (X, d) be a metric space and A, B be two closed and disjoint subset of X. Then there exists two open sets $G, H \ni A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$.



proof:

If either A or B is empty, ($A = \emptyset$ or $B = \emptyset$) say A is empty ($A = \emptyset$) then $X \cap \emptyset = \emptyset$ and $A \subseteq A$ and $B \subseteq X$ and we are done.

On the other hand, if $A, B \neq \emptyset$ and $a \in A$ then $a \notin B$. Since B is closed then from Corollary 4.4.2 $d(a, B) = \delta_a > 0$.

Similarly, if $b \in B$ then $b \notin A$ and $d(b, A) = \delta_b > 0$.

We now set an open sphere

$S_a = S(a, \delta_a/3)$ and $S_b = S(b, \delta_b/3)$

i.e. $a \in S_a$ and $b \in S_b$. We claim that

$G = \bigcup\{S_a \ni a \in A\}$ and $H = \bigcup\{S_b \ni b \in B\}$ are disjoint open sets containing A and B respectively.

Clearly, G and H are open and they contain A and B respectively. So we only need to show that $G \cap H = \emptyset$.

Suppose if possible, $G \cap H \neq \emptyset$ and so let $p \in G \cap H \Rightarrow \exists\, a_0 \in A$ and $b_0 \in B \ni$

$p \in S_{a_0}$ and $p \in S_{b_0}$. Let $d(a_0, b_0) = \varepsilon > 0$

Then $d(a_0, B) = \delta_{a_0} \leq \varepsilon$ and $d(b_0, A) = \delta_{b_0} \leq \varepsilon$

However $p \in S_{a_0}$ and $p \in S_{b_0}$ and so

$\therefore d(a_0, p) < \delta_{a_0}/3$ and $d(b_0, p) < \delta_{b_0}/3$.

Then by triangular inequality of the metric,

$\varepsilon = d(a_0, b_0) \leq d(a_0, p) + d(p, b_0) < \delta_{a_0}/3 + \delta_{b_0}/3$

$\leq \varepsilon/3 + \varepsilon/3$

$= 2\varepsilon/3$

This is clearly a contradiction and so

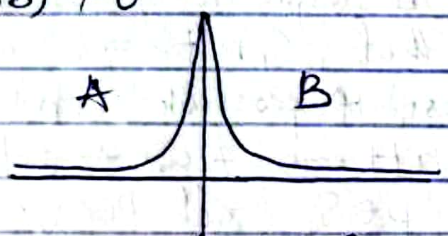
$G \cap H = \emptyset$

Afterstrike

11/1/23

Remark 4.5.1 : One would expect from theorem 4.5 that the distance between any two disjoint closed sets is always greater than zero. This is however not always true as the next example shows.

Ex : Consider the following two sets on the $\mathbb{R}^2$ plane
$A = \{(x,y) : xy \geq 1,\ x < 0\}$, $B = \{(x,y) : xy \geq 1,\ x > 0\}$.
A and B are both closed and also disjoint, but $d(A,B) \neq 0$



DEF 4.6 : Let (X, τ) be a topological space. The space is said to be metrizable if there exists a metric on X which induces a topology τ.

Ex 1. The discrete space (i.e the space with the discrete topology) is metrizable and the topology is induced by the trivial metric

2. The usual space i.e the topological space with usual topology is metrizable and the topology is induced by the usual metric.

4.6.1 : The problem of finding the necessary and sufficient condition, for a topological space to be metrizable is called the metrization problem.

Exercise

Let (X, d) be a metric space. Show that if $A, B \subseteq X$ then i- $d(\bar{A}) = d(A)$

ii- $d(A \cup B) = d(A) + d(B) + d(A \cap B)$.

5 CHAPTER FIVE

SEPARATION AXIOMS

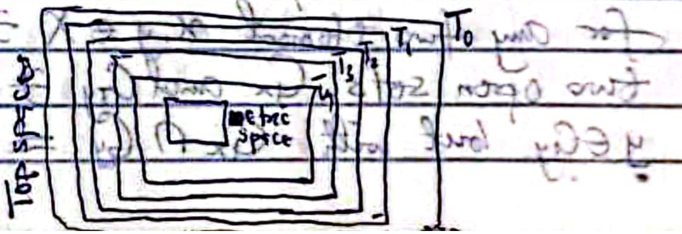
DEF 5.1 : A topological space (X, τ) is said to be a T_0 - space if for any two points $x, y \in X$, with $x \neq y$ $\exists$ at least one open set G containing one of the elements and not the other.

Ex If $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$E_n = \{n, n+1, n+2, n+3, \ldots\}$

$\tau = \{E_1, E_2, E_3, \ldots\}, \emptyset$

Then $(\mathbb{N}, \tau)$ is a T_0 - space. Also the usual space $(\mathbb{R}, U)$ is also a T_0 - space. However the indiscrete space (X, d) is not a T_0 - space because if X has two or more elements, the condition of T_0 cannot be satisfied.



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DEF 5.2 A topological space (X, τ) is said to be T_1 space if for any two elements $x, y \in X$, with $x \neq y$, $\exists$ two open sets G_x, G_y where $x \in G_x$ but $y \notin G_x$ and $y \in G_y$ but $x \notin G_y$.

EX 1. The topological space $(\mathbb{N}, \tau)$ defined above is not a T_1 space, even though it is a T_0-space.

2. The usual space $(\mathbb{R}, U)$ is a T_1-space.

3. The cofinite space is a T_1 space because if $x, y \in X \ni x \neq y$ then $\{x\}^c$ and $\{y\}^c$ are open sets $\ni y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$

$\Rightarrow$ The cofinite space is a T_1-space

This cofinite topology is the coarsest T_1-topology which is why the cofinite topology is sometimes referred to as T_1-topology.

Note that, $T_1 \Rightarrow T_0$ but there are T_0-spaces which are not T_1-space, such as $(\mathbb{N}, \tau)$

DEF 5.3 A topological space (S, τ) is said to be a T_2-space or Hausdorff space if for any two element $x, y \in X \ni x \neq y$, $\exists$ two open sets G_x and $G_y \ni x \in G_x$ and $y \in G_y$ but with $G_x \cap G_y = \emptyset$.

17

EX 1- The usual space $(\mathbb{R}, U)$ is a T_2-space.

2- The discrete space (X, D) is also T_2-space

Note also that $T_2 \Rightarrow T_1 \Rightarrow T_0$

3- A cofinite space may or may not be a T_2-space or Hausdorff space. When X contains an infinite number of elements, then $(X, \mathfrak{J})$ cofinite space is not a T_2-space, because suppose $\exists$ $x, y \in X \ni x \neq y$ and G_x, G_y are the two open sets $\ni x \in G_x$, $y \in G_y$ and $G_x \cap G_y = \emptyset$

$\Rightarrow G_x \subseteq G_y^c$ or $x \in G_x \subseteq G_y^c$

which is a contradiction since $|G_x|$ is infinite and $|G_y^c|$ finite.

$\Rightarrow G_x \cap G_y \neq \emptyset$ and so the space is not T_2.

Note again that any metric space is a T_2-space and so it is T_1-space and consequently a T_0-space

DEF 5.4: A topological space (X, τ) is said to be regular if for $x, y \in X \ni x \neq y$ we have $x \in X$, $x \notin F$ and F is a closed set $\ni F \subseteq X$ $\exists$ two open sets G_x and $G_F \ni x \in G_x$, $F \subseteq G_F$ with $G_x \cap G_F = \emptyset$.

EX 1- The usual space $(\mathbb{R}, U)$ is a regular space

2- The indiscrete space is also automatically regular because it does not have any open or close sets

$\Rightarrow \{p\}^c = \cup \{G_x \ni x \in \{p\}^c\}$

$\Rightarrow \{p\}^c$ is open since it is a union of open sets.

$\therefore \{p\}$ is closed and the lemma is proved.

18/1/23

THEOREM 5.4.3: Every T_3-space is also a T_2-space

~~proof~~:

Every T_3-space is also T_1 just like any T_2-space. Now we want to show that for $x \neq y$ $\exists$ two open sets $G_x, G_y \ni x \in G_x, y \in G_y$ and $G_x \cap G_y = \phi$.

To do that, we know from lemma 5.4.2 that every singleton set is closed in a T_1-space. So without any loss of generality we may let $\{x\}$ be closed. Then $y \notin \{x\}$ and $\{x\}$ is closed. So by T_3 $\exists$ two open sets G_x and $G_y \ni \{x\} \subseteq G_x$ and $y \in G_y$ with $G_x \cap G_y = \phi$.

$\Rightarrow$ The space is T_2

DEF 5.5: A topological space $(X, \mathcal{T})$ is said to be normal if for any closed sets F_1 and $F_2 \subseteq X$ such that $F_1 \cap F_2 = \phi$, $\exists$ two open sets $G_1, G_2 \ni F_1 \subseteq G_1$ and $F_2 \subseteq G_2$ with $G_1 \cap G_2 = \phi$

THEOREM 5.1: Every metric space (X, d) is normal.

proof:

This is true from theorem A.5

DEF 5.5.2: The normal T_1-space is called a T_4-space.

THEOREM 5.5.3: Every T_4-space is also a T_3-space

proof:

This is true because if (X, τ) is a T_4-space then it is a T_1-space and normal

$\Rightarrow$ for any closed sets $F_1, F_2 \ni F_1 \cap F_2 = \phi$ $\exists$ two open sets G_1, G_2 with $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$ where $G_1 \cap G_2 = \phi$.

However every T_3-space is also T_1-space. We only need to show regularity. To do that let $F \subseteq X$ and $x \notin F$. Since X is T_1, then every singleton set is closed from terms 5.4.2. Therefore $\{x\}$ is also closed

$\Rightarrow F \cap \{x\} = \phi$ because $x \notin F$. By normality, $\exists$ two sets $G_1, G_2 \ni F \subseteq G_1$ and $\{x\} \subseteq G_2$ with $G_1 \cap G_2 = \phi$.

$\Rightarrow X$ is also regular

$\Rightarrow X$ is T_1 and regular so it is T_3.

Let τ is a topology generated by closed set.

THEOREM 5.5.4: Every metric space is a T_4-space.

proof: exercise

EX: Consider $X = \{a, b, c\}$ and suppose $\tau_1 = \{\phi, X, \{a\}, \{b, c\}\}$. Then (X, τ_1) is both regular and normal.

solution:

Closed sets $= \{X, \phi, \{b, c\}, \{a\}\}$ — All are complement

$a \notin \{b, c\}$ $\quad a \notin \{\phi\}$ $\quad b \notin \{a\}$

$\{a\} \cap \{b, c\} = \phi$ $\quad \{a\} \cap \{\phi\} = \phi$ $\quad \{b, c\} \cap \{a\} = \phi$

On the other hand if $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, then the ~~closed sets of~~ (X, τ_2) is normal but it is not regular.

solution:

Closed sets $= \{X, \phi, \{b, c\}, \{a, c\}, \{c\}\}$

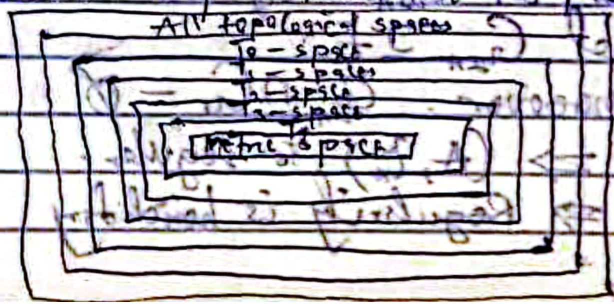
$a \notin \{\phi\}$

$c \notin \{\phi\}$

$a \notin \{b, c\}$ is not possible for the case $X \cap \phi$

$\therefore (X, \tau_2)$ is not regular. However it is not normal

(*) Relationship Between the T_i spaces



24/1/20

6: A property of a topological space is said to be hereditary if the property holds for every sub-space.

The property of being a T_0-space, T_1-space, T_2-space, T_3-space, T_4-space, regularity is Hereditary.

On the other hand, however the property of being a T_0-space, T_1-space, T_2-space, T_3-space, T_4-space, regularity, normality is topological.

Exercise

prove that the property of being T_0, T_1, T_2, and T_3 is both topological and hereditary.

To show that regularity is hereditary

Consider a closed set F_A in A where $F_A = F \cap A$ for some closed set $F \subseteq X$

Let $x \notin F_A \Rightarrow x \notin A$.

Since (X, τ) is regular and $x \notin F$, then $\exists\, G_1, G_2 \ni x \in G_1$ and $F \subseteq G_2$ with $G_1 \cap G_2 = \phi$.

Then let $G_{2A} = G_2 \cap A$ which is open in A and $G_{2A} \supseteq F_A$.

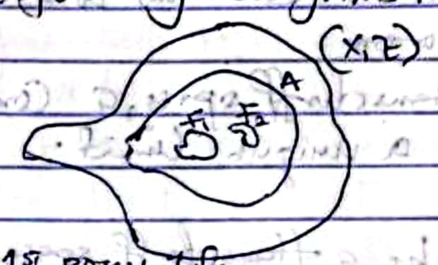
moreover $G_{2A} \cap G_1 = \emptyset$

$\Rightarrow (A, \tau_A)$ is regular.

$\Rightarrow$ Regularity is hereditary

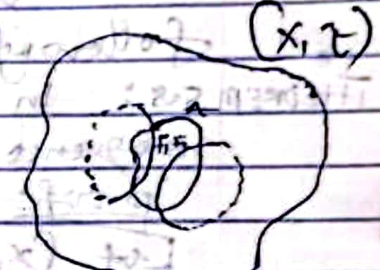
To show that normality is not hereditary, consider the following diagrams:

(i)



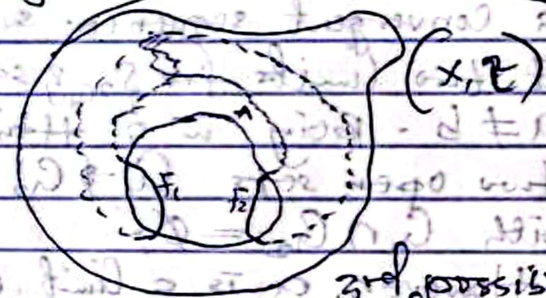
1st possibility

(ii)



2nd possibility

(iii)



3rd possibility

DEF 5.7: Let (X, τ) be a topological space. A sequence $\{x_n\}$, where $x_n \in X$ is said to converge to an element $x \in X$ if every open set G containing x, $\exists\, N \in \mathbb{N} \ni x_n \in G\ \forall n \geq N$.

Example

1) In a discrete space is not possible for the sequence to converge.

2) In a co-finite space where the number of elements is infinite, every element is a limit point.

3) In a Hausdorff space, convergent seq. has a unique limit. This lead to the following theorem.

*THEOREM 5.8: In a Hausdorff space, a convergent sequence has a unique limit.

proof:

Let (X, τ) be a Hausdorff space and $\{x_n\}$ a convergent sequence. Suppose if ~~possible~~ $\exists$ two limits for $\{x_n\}$ say a and b where $a \neq b$. Being in a Hausdorff space $\exists$ two open sets G_1 & $G_2 \ni a \in G_1$ and $b \in G_2$ with $G_1 \cap G_2 = \emptyset$.

However, a is a limit of $\{x_n\}$ and $a \in G_1$

$\Rightarrow \exists N_1 \in \mathbb{N} \ni x_n \in G_1 \; \forall n \geq N_1$.

Similarly since b is also a limit of $\{x_n\}$ and $b \in G_2$

$\Rightarrow \exists N_2 \in \mathbb{N} \ni x_n \in G_2 \; \forall n \geq N_2$

If we choose $n \geq \max(N_1, N_2)$ then

$x_n \in G_1$ and $x_n \in G_2$

$\Rightarrow x_n \in G_1 \cap G_2$. This is a contradiction since ~~$x_n \in G$~~ $G_1 \cap G_2 = \emptyset$

$\Rightarrow$ We cannot have two limits in Hausdorff.

25/1/23

*THEOREM 5.9: A topological space is normal iff for any closed set F and an open set $G \ni F \subseteq G$ $\exists$ an open set $H \ni F \subseteq H \subseteq \bar{H} \subseteq G$

proof

Suppose (X, τ) is a normal topological space and $F \subseteq G$ where F is closed and G is open i.e $G \in \tau$. G^c is closed. so let $F_1 = G^c \subset F^c$

$\Rightarrow F \cap F_1 = \emptyset$.

$\Rightarrow \exists$ two open sets G_1 and $G_2 \ni F \subset G_1, F_1 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$. $\Rightarrow G_1 \subseteq G_2^c$

$\Rightarrow F \subseteq G_1 \subseteq G_2^c$

$\Rightarrow F \subseteq G_1 \subseteq \bar{G}_1 \subseteq G_2^c$

$\Rightarrow F \subseteq G_1 \subset \bar{G}_1 \subseteq G$ since $G_2^c \subseteq F_1^c \subseteq G$

$\Rightarrow F \subseteq G_1 \subset \bar{G}_1 \subset G$. Now let $H = G_1$

$\Rightarrow F \subseteq H \subseteq \bar{H} \subseteq G$ as required

Conversely, if (X, τ) is a top space $\ni$ for any closed set $F \subseteq G$ where $G \in \tau$, $\exists H \in \tau \ni F \subseteq H \subset \bar{H} \subseteq G$ as defined. let F_1, F_2 be closed sets in $X \ni F_1 \cap F_2 = \emptyset \rightarrow F_1 \subset F_2^c$ where F_2^c is open. so let $F_2^c = G \Rightarrow F_1 \subseteq G$

$\Rightarrow \exists H \subset \tau \ni F_1 \subset H \subseteq \bar{H} \subseteq G = F_2^c$. If we let $G_1 = H$, $G_2 = \bar{H}^c$ then $F_1 \subseteq G_1$ and $G_2 = \bar{H}^c \supset (F_2^c)^c$

$\Rightarrow F_2 \subseteq \bar{H}^c = G_2$

Lastly $G_1 \cap G_2 = H \cap \bar{H}^c \subseteq H \cap H^c = \emptyset$

$\therefore (X, \tau)$ is normal. $\square$

5.10: (Urysohn's lemma)

Let F_1 and F_2 be disjoint closed subsets of a normal space X. Then $\exists$ a continuous function $f: X \to [0,1] \ni f(F_1) = \{0\}$ and $f(F_2) = \{1\}$

proof:

Consider the dyadic fractions in D (dyadic numbers) contained in the set D. i.e D contains fractions whose denominators are powers of two in the unit interval $[0,1]$

Thus

$$D = \left\{ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \tfrac{1}{16} \cdots \right\}$$

so every point of D is a limit point of D.

By hypothesis $F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c$ and from theorem 5.9 $\exists$ an open set $G_{1/2} \ni$

$$F_1 \subseteq G_{1/2} \subseteq \bar{G}_{1/2} \subseteq F_2^c$$

By the same theorem $\exists$ an open set $G_{1/4} \ni F_1 \subseteq G_{1/4} \subseteq \bar{G}_{1/4} \subseteq G_{1/2} \subseteq \bar{G}_{1/2} \subseteq F_2^c$

so continuing in this way, for each $t \in D$, $\exists$ an open set $G_t \ni$ if $t_1, t_2 \in D$ and $t_1 < t_2$ then $\bar{G}_{t_1} \subseteq G_{t_2}$. Define a function $f(x)$ as follows:

$$f(x) = \begin{cases} \inf(t) \ni x \in G_t & \text{if } x \notin F_2 \\ 1 & \text{if } x \in F_2 \end{cases}$$

Then for every $x \in X$, $0 \leq f(x) \leq 1$ and from definition $f(F_2) = \{1\}$.

Moreover, since $F_1 \subseteq G_t \; \forall \; t$ then for any $x \in F_1$, $f(x) = 0 \Rightarrow f(F_1) = \{0\}$.

We therefore just need to show that f is continuous and the lemma is prove.

To show the continuity, we need to show that $f^{-1}[0,a]$ and $f^{-1}[b,1]$ where $a, b \in [0,1]$ are open subset of X.

(prove it as exercise) see the chapter on T-space on Stcherm's series

To prove first that $f^{-1}([0,a))$ is open, let $x \in f^{-1}[0,a)$ then $f(x) \in [0,a)$ i.e $0 \leq f(x) < a$

Since D is dense in $[0,1]$, $\exists$ $t_x \in D \ni f(x) < t_x < a \Rightarrow f(x) = \inf\{t \ni x \in G_t\} < t_x < a$

Accordingly $x \in G_{t_x}$ where $t_x < a$. ~~then it~~ $x \in \bigcup \{G_t \ni t < a\}$

$$\Rightarrow f^{-1}[0,a) \subseteq \bigcup \{G_t \ni t < a\}$$

On the other hand, if $y \in \bigcup\{G_t \ni t < a\}$, then $\exists\ t_y \in D \ni t_y < a$ and $y \in G_{t_y}$

$\therefore f(y) = \{\inf(t) \ni y \in G_t\} \leq t_y < a$

$\Rightarrow y \in f^{-1}[0,a)$ and so

$\bigcup\{G_t \ni t < a\} \subseteq f^{-1}[0,a)$

Hence $f^{-1}[0,a) = \bigcup G_t \ni t < a$

and so $f^{-1}[0,a)$ is open.

In a similar way, it can be shown (show it) that $f^{-1}(b,1]$ is also open.

3/11/23

DEF 5.8: A topological space (X,τ) is said to be completely regular, if for any closed set F and $x \notin F$, $\exists$ a continuous function $f: X \to [0,1]$ $\ni f(x) = 0$ and $f(F) = \{1\}$.

Exercise

i) show that complete regularity is both topological and hereditary.

ii) prove that a complete regular space is also regular

6 CHAPTER SIX

COUNTABILITY AXIOMS

DEF 6.1: A topological space (X,τ) is said to satisfy the first axiom of ~~Countability~~ Countability if it has a countable local base at each point $x \in X$ i.e at each point $x \in X$, $\exists$ a countable subset $\beta^x = \{B_1^x, B_2^x, \ldots\}$ where $B_i^x \in \tau \ni$ for any $G \in \tau$ with $x \in G$ $\exists$ $B_i^x \ni x \in B_i^x \subseteq G$.

A topological space is sometimes referred to as a first countable space or a first axiom space or a C_1-space.

Examples:

i) The usual space $(\mathbb{R}, U)$ is a first axiom space and the local base is $\beta^x = \{(a,b) \ni a < x < b,\ a,b \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rational numbers.

ii) Every discrete space is a first axiom space and the local base is $\beta^x = \{\{x\}\}$ where $\{x\}$ is a singleton

iii) Every metric space (X,d) is a first axiom space and the local base is $\beta^x = \{S(x, 1/n)\}$ which is countable since $n \in \mathbb{N}$ which is countable

iv) The cofinite is not a first axiom space if X is uncountable. To see this let $x \in X$ and if possible $\exists$ a countable local $\beta^x = \{B_1, B_2 \ldots$

then $B_n \in \tau$ and B_n^c in finite we write
$A = \bigcup_{n=1}^{\infty} B_n^c$, then A is countable and A^c is uncountable.
Also $x \in \bigcap_{n=1}^{\infty} A$ and since $A = \bigcup P_n^c$

$\Rightarrow A^c = \left(\bigcup_{n=1}^{\infty} B_n^c\right)^c = \bigcap_{n=1}^{\infty} B_n \Rightarrow x \in A^c$. Choose $y \in A^c$ but $x \neq y$. Then let $G = \{y\}^c$

$\Rightarrow G$ is open in (X, τ) and $x \in G$

$\Rightarrow \exists$ an open set $B_k \ni x \in B_k \subseteq G$.

$\Rightarrow B_k \subseteq \{y^c\}_0 \Rightarrow y \notin B_k$.

However $y \in A^c = \bigcap_{n=1}^{\infty} B_n$ which is a contradiction since $y \in \bigcap_{n=1}^{\infty} B_n$ therefore (X, τ) is not a C_1-space.

DEF 6.2: A topological space (X, τ) is said to satisfy the second axiom of countability if the topology τ has a countable base. The topology is then called a second axiom space or second countable space or a C_2-space.

Example

i) The usual space $(\mathbb{R}, \mathcal{U})$ is a second countable space and the base for τ is $\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rational numbers. However it will not be a second countable space if $\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R}\}$ since $\mathbb{R}$ is not countable.

ii) A metric space (X, d) may or may not be a second countable space. This is the case when X is uncountable and d is the trivial metric then the metric will induce the discrete topology. The base in that case is $\mathcal{B} = \{\{x\} \mid x \in X\}$ and this is not a countable base.

Note that the second axiom space is necessarily a first axiom space but not vice-versa. This is true since if $\mathcal{B} = \{B_1, B_2, \ldots\}$ is a base for τ and $\mathcal{B}^x = \{B_1^x, B_2^x, \ldots\}$ is a local base at x then $\mathcal{B}^x \subseteq \mathcal{B}$.

Separable top. space

1/2/23

DEF 6.3 A topological space (X, τ) is said to be separable if it has a countable dense subset i.e. if $\exists$ $A \subseteq X \ni \bar{A} = X$ and A is countable.

Example

1. The usual space $(\mathbb{R}, \mathcal{U})$ is separable and the dense countable subset in $\mathcal{U}$ is the set $\mathbb{Q}$ of rationals i.e. $\bar{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable.

2. The discrete space $(\mathbb{R}, \mathcal{D})$ is not separable because every singleton set is an open set and so the only dense subset $\mathbb{R}$ in that case is the $\mathbb{R}$ itself and $\mathbb{R}$ is not countable.

*THEOREM 6.4: Let (X,τ) be a second countable space. Then it is separable.

proof:

Let (X,τ) be a second countable space (C_2) and $\mathcal{B}=\{B_1,B_2,\dots\}$ be a countable base for τ. Let A be a subset of X consisting of one element from each of the set B_k's.

Then clearly A is countable. We now show that $\bar{A}=X$ and we are done. Consider one element $x\in A^c$ but $x\in X$. Then for any $G\ni x$ where $G\in\tau$ $\exists$ an open set $B_k\in\mathcal{B}\ni x\in B_k\subseteq G$. From the definition of A,

$\Rightarrow A\cap B_k\neq\emptyset$

Since $x\notin A$ then $[A\cap B_k\setminus\{x\}]\neq\emptyset$.

$\Rightarrow x\in A'$ and so $A^c\subseteq A'$

$\therefore X=A\cup A^c\subseteq A\cup A'=\bar{A}$

$\Rightarrow A$ is dense in X. □

Note that the converse of the theorem is not always true i.e. there could be separable topological space which is not C_2. On the other hand metric space which is separable is C_2 as we see in the next theorem.

*THEOREM 6.5: A separable metric space is C_2.

proof:

Let (X,d) be a separable metric space i.e. it has a dense countable subset A. Consider $\mathcal{B}=\{S(a,\delta)\}$ $\ni a\in A$, $\delta>0$ and $\delta\in\mathbb{Q}^+$. Clearly $\mathcal{B}$ is countable since $\mathbb{Q}^+$ is countable. Let $x\in X$ and $G\in\tau$ $\ni x\in G$. Since G is open, $\exists\ \epsilon>0\ni S(x,\epsilon)\subseteq G$

$\bar{A}=X\Rightarrow x\in A$ or $x\in\bar{A}$.

If $x\in A$, then $S(x,\delta)$ where δ is rational and $\delta<\epsilon$ will be an element of $\mathcal{B}$ contained in G.

On the other hand if $x\notin A\Rightarrow x\in\bar{A}$.

$\Rightarrow\exists\ a\in A\ni d(x,a)<\epsilon/3$ i.e.

$a\in S(x,\epsilon/3)$. Let δ be a rational number

$\ni\epsilon/3<\delta<2/3$, then

$S(a,\delta)\subseteq G$ and $x\in S(a,\delta)$,

Since $d(x,a)<\epsilon/3<\delta$.

for any $y\in S(a,\delta)$

$\Rightarrow d(a,y)<\delta$ and

$d(x,y)\leq d(x,a)+d(a,y)$

$<\epsilon/3+\delta$

$<\epsilon/3+2/3\,\epsilon=\epsilon$

$\Rightarrow y\in S(x,\epsilon)\subseteq G$

$\therefore S(a,\delta)\subseteq S(x,\epsilon)\subseteq G$ which follows that $\mathcal{B}$ is a countable for (X, d) and so (X, d) is second countable.

From the foregoing, it is clear that

$$C_1 \Leftarrow C_2 \Rightarrow \text{separable and metric space} + \text{separable} \Rightarrow C_2.$$

THEOREM 6.6: (Urysohn's metrization theorem)
Every second countable normal T_1 topological space is metrizable.

proof exercise

7/2/23

CHAPTER SEVEN
COMPACTNESS

DEF 7.1 i) Let (X, τ) be a topological space and $A \subseteq X$. A class $\mathcal{G}$ of open sets of X is called an open cover of A, if $A \subseteq \mathcal{G} = \{\cup G \ni G \in \mathcal{G}\}$

DEF 7.1 ii) If $\mathcal{G}_1 \subset \mathcal{G} \ni \mathcal{G}_1$ is also an open cover of A then $\mathcal{G}_1$ is called a sub-cover of A. let

$$\mathcal{G}_1 = \{G_i \in \tau \ni A \subseteq \cup G_i\}.$$

DEF 7.1 iii) If in particular, $|\mathcal{G}_1|$ is finite, then $\mathcal{G}_1$ is called a finite subcover.

DEF 7.2: Let (X, τ) be a topological space and $A \subseteq X$. A is said to be compact if every open cover of A can be reduced to a finite sub cover of A.

28

Example i:
A closed and bounded interval $[a, b]$ of $\mathbb{R}$ with the usual topology is compact (this is base on the Heine-Borel-theorem)

Example ii:
Clearly any finite set is compact with respect to any topology.

Example iii)
If (X, τ) is a discrete space and X is infinite, then (X, τ) is not compact

Example (iv)
If (X, τ) is the co-finite space where $A \subseteq X$ and A is infinite i.e $|A| =$ infinite then A is compact. This is because if $A \subseteq \{\bigcup_{i \in I} G_i \ni G_i \in \mathcal{G}\}$ then $A - G_1 = A \cap G_1^c$ is finite and so, choose one open set containing each element of $A - G_1$.

Consequently, $A = (A - G_1) \cup G_1$

$\Rightarrow A \subseteq \bigcup_{i=1}^{n} H_i \cup G_1$ where $H_i \in \tau$ which is certainly a finite open cover of A.

$\therefore$ A is compact.

where $H_i = Y \cap G_i$ and $H_i \in \tau_Y$.

Since ~~from~~ A is compact in (Y, τ_Y) then $\exists$ a finite number of H_i's say $H_1, H_2, \ldots, H_r$ $\ni A \subseteq \bigcup_{i=1}^{r} H_i \Rightarrow A \subseteq \bigcup_{i=1}^{r} (Y \cap G_i)$

$\Rightarrow Y \cap \bigcup_{i=1}^{r} G_i \Rightarrow A \subseteq Y \cap \bigcup_{i=1}^{r} G_i \Rightarrow A \subseteq \bigcup_{i=1}^{r} G_i$

$\Rightarrow A$ is compact in (X, τ).

Theorem 7.4: A continuous image of a compact set is compact.

proof:

Let $f : (X, \tau) \to (Y, \tau^*)$ be a continuous function. Let A be a compact set in (X, τ). Consider an open cover for $f(A) \ni f(A) \subseteq \bigcup_{i \in I} \{H_i \ni H_i \in \tau^*\}$

Then $A \subseteq f^{-1}(\bigcup_{i \in I} H_i) = \bigcup_{i \in I} f^{-1}(H_i)$. Since f is continuous and H_i is open then $f^{-1}(H_i)$ is also open in (X, τ). Since A is compact then $\exists$ a finite number of $f^{-1}(H_i)$ that will cover A i.e.

$A \subseteq f^{-1}(H_{i_1}) \cup f^{-1}(H_{i_2}) \cup \ldots \cup f^{-1}(H_{i_k}) = f^{-1}(H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_k})$

$\Rightarrow f(A) \subseteq H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_k} = \bigcup H_{i_k}$

$\Rightarrow f(A)$ is compact. $\square$

Remark 7.4.1:

Compactness is a topological property

*THEOREM 7.5: A closed subset F of a compact space (X, τ) is compact.

proof:

Let (X,τ) be a compact topological space and F a closed subset of X. Let $F \subseteq \bigcup_{i\in I} G_i$ where $G_i \in \tau$ then $X = F \cup F^c \Rightarrow X = \bigcup_{i\in I} G_i \cup F^c$ which is an open cover for X.

However, X is compact

$\Rightarrow \exists$ a finite subcover for X i.e $\exists$ $r \ni$ $X = \bigcup_{i=1}^{r} G_i \cup F^c \Rightarrow F \subseteq \bigcup_{i=1}^{r} G_i$

$\Rightarrow F$ is compact. □

note that: $F \cap F^c = \emptyset$

*THEOREM 7.6: A compact subset of a Hausdorff space is closed.

proof:

Let (X,τ) be a Hausdorff space and $A \subseteq X$ be compact. Let $x \in A^c$ then for every $a \in A$, $x \neq a$ and since we are in a T_2-space, $\exists$ an open sets H_a and $G_a \ni x \in H_a, a \in G_a$ and $H_a \cap G_a = \emptyset$.

$\therefore A \subseteq \bigcup_{i\in I} \{G_{a_i} \ni G_{a_i} \in \tau\}$. Since A is compact, $\exists$ a finite number of sets $G_{a_1}, G_{a_2}, \ldots G_{a_k} \ni A \subseteq \bigcup_{i\in I} G_{a_i} = G$ say

write $H = H_{a_1} \cap H_{a_2} \cap \ldots \cap H_{a_k}$ where $x \in H_{a_i}$ and $H_{a_i} \cap G_{a_i} = \emptyset$

$\Rightarrow A \subseteq G$ and $x \in H$ where

$G \cap H = \bigcup_{i=1}^{k} G_{a_i} \cap \left(\bigcap_{i=1}^{k} H_{a_i}\right)$

$= \bigcup_{i=1}^{k} \{G_{a_i} \cap H\}$

However $G_{a_i} \cap H \subseteq G_{a_i} \cap H_{a_i} = \emptyset$

$\Rightarrow H \cap A = \emptyset$

$\Rightarrow A^c = \{x / x \in A^c\} \subseteq \bigcup_{x\in A^c} \subseteq A^c$

$\Rightarrow A^c = \bigcup\{H \ni H \in \tau\}$

$\Rightarrow A^c$ is open and so A is closed. □

Remark 7.6.1

Suppose (X,τ) is a Hausdorff space. If $x \in X$, $A \subseteq X \ni A$ is compact and $x \notin A$, then $\exists$ open sets $H, G \ni x \in H$, $A \subseteq G$ and $H \cap G = \emptyset$.

*THEOREM 7.7: Let (X,τ) be a Hausdorff space. Suppose A and B are two compact subsets of X such that $A \cap B = \emptyset$. Then there exists two open sets H and $G \ni A \subseteq H$ and $B \subseteq G$ where $H \cap G = \emptyset$.

Proof

~~Let (X,τ) be a T_2-space~~

pto

Proof:

For each $b \in B$, $b \notin A$ and A is compact
$\Rightarrow \exists$ open sets $G_b, H_b \ni b \in G_b$ and $A \subseteq H_b$ with $G_b \cap H_b = \emptyset$ from remark 7.6.1

$B = \{b \mid b \in B\} \subseteq \bigcup_{b \in B} \{G_b \mid G_b \in \tau\}$

Since B is compact and $\bigcup_{b \in B} \{G_b \mid G_b \in \tau\}$ is an open cover for B then $\exists$ a finite number of G_b's say

$G_{b_1}, G_{b_2}, \ldots G_{b_k} \ni B \subseteq G_{b_1} \cup G_{b_2} \cup \ldots \cup G_{b_k} = \bigcup_{i=1}^{k} G_{b_i}$

write $H = H_{b_1} \cap H_{b_2} \cap \ldots \cap H_{b_k}$ where

$$G_{b_i} \cap H_{b_i} = \emptyset$$

Let $G = \bigcup_{i=1}^{k} G_{b_i} \Rightarrow B \subseteq G$ and $A \subseteq H$

Lastly: $G \cap H = \bigcup_{i=1}^{k} G_{b_i} \cap H = \bigcup_{i=1}^{k} (G_{b_i} \cap H) = \emptyset$

since $G_{b_i} \cap H \subseteq G_{b_i} \cap H_{b_i} = \emptyset$.

COROLLARY 7.7.1

A compact Hausdorff space is normal and so it is T_4.

Proof as exercise

DEF 7.8: Let (X, τ) be a topological space. A subset of X i.e. $A \subseteq X$, ~~A~~ is said to be countably compact, if every infinite subset of it has a limit point in A.
(Remember Weierstrass Bolzano's theorem).

THEOREM 7.9 *

A compact subset is countably compact.

Proof:

~~Suppose A is compact~~ ~~Countable~~ ~~Compact~~

Let A be a compact subset of a topological space (X, τ). Let B be any infinite subset of A and that B does not have any limit point in A. That is no point of A (including those in B) is a limit point of B.

Thus for any $x \in A$, $\exists$ an open set $G_x \ni x \in G_x$ and $(B \cap G_x) \setminus \{x\} = \emptyset$. Then $A \subseteq \bigcup_{x \in A} G_x$

$\Rightarrow G_x$ has at most one element of B.

The union $\bigcup_{x \in A} G_x$ is an open cover of A and A is compact so $\exists$ a finite number of the G_x's say $G_{x_1}, G_{x_2}, \ldots G_{x_k} \ni$

$A \subseteq \bigcup_{i=1}^{k} G_{x_i}$. However $|B \cap G_{x_i}| = 0$ or 1 for $i = 1, 2, 3, \ldots k$.

$\Rightarrow |B| \leq k$

$\Rightarrow B$ is finite

which is a contradiction since B is infinite from the hypothesis.

Therefore B must have limit point

$\Rightarrow A$ is countably compact.

Remark 7.9.1

The Converse of the theorem does not necessarily hold because if $N=\{1,2,3,\dots\}$ and $B=\{\{1,2\},\{3,4\},\{5,6\},\dots\}$ is a base for some topology on N. Let A be a non-empty subset of N say $n_0 \in A$. If n_0 is odd then n_0+1 is a limit point of A and if n_0 is even then n_0-1 is a limit point of A. In either case, therefore A will have a limit point so (N,τ) is Countably compact. However $A=\{\{1,2\},\{3,4\},\{5,6\},\dots\}$ is an open cover of N but there is no finite subcover i.e there is no proper subset of A which covers N.

THEOREM 7.10: 15/2/23

A Compact subset of a metric space is bounded.

proof:

Let (X,d) ~~be a metric space~~ and A a compact subset of X. Consider the open spheres $\{S(x,1) / x\in A\}$. These form an open cover for A. Since A is compact, $\exists$ a finite number of these open spheres which cover A. i.e

$$A \subseteq S(x_1,1)\cup S(x_2,1)\cup\dots S(x_k,1)$$

$$\text{or } A\subseteq \bigcup_{i=1}^{k} S(x_i,1):$$

If $x,y\in A$ then

$$d(x,y)\le d(x,x_i)+d(x_i,x_j)+d(x_j,y)$$

Where $x\in S(x_i,1)$ and $y\in S(x_j,1)$

$$\le 1+2k+1<\infty$$

let $1+2k+1=M$

$d(x,y)\le M \Rightarrow A$ is bounded. □

DEF 7.11: A collection $A=\{A_i \ni A_i\in X\}$ is said to have finite Intersection property if the Intersection of any finite number of the set $A_i\in A$ is not empty.

Example (1)

The set (n,∞) for $n=1,2,3,4\dots$ has a finite Intersection property.

for Intersection if we Intersect

$(3,\infty),(4,\infty),(20,\infty),(105,\infty),(1000,\infty)$

$$(3,\infty)\cap(4,\infty)\cap(20,\infty)\cap(105,\infty)\cap(1000,\infty)=(1,000,\infty)$$

This implies if we pick $(a_1,\infty),(a_2,\infty)\dots(a_k,\infty)$

Then the Intersection is (a_s,∞) When $a_s\ge a_i$, $i=1,2,\dots k$.

Example (2)

The set $(0, \frac{1}{n})$ for $n = 1, 2, 3, 4 \ldots$

for Instance if we Intersect

$(0, \frac{1}{8}), (0, \frac{1}{3}), (0, \frac{1}{20}), (0, \frac{1}{50}), (0, \frac{1}{102})$

$(0, \frac{1}{8}) \cap (0, \frac{1}{3}) \cap (0, \frac{1}{20}) \cap (0, \frac{1}{50}) \cap (0, \frac{1}{102})$

$= (0, \frac{1}{102})$

This Implies if we pick, $(0, a_1)(0, a_2) \ldots (0, a_k)$

then the Intersection is

$(0, a_s)$ where

$a_s \leq a_i$, for $i = 1, 2, 3 \ldots k$

NOTE THAT; If the Infinite Intersection is non-empty then authomatically, the finite Intersection will also be empty.

However the Converse is not necessarily true as we have in the Case of last two examples.

21/2/23

THEOREM 7.12

A topological space (X, τ) is compact iff every collection of closed set with finite Intersection property has a non-empty Intersection itself.

proof:

Suppose (X, τ) is compact and $\{F_i\}$ is a collection of closed sets such that

$\ni \bigcap_{i \in I} F_i = \emptyset$ then we need to show that $\bigcap F_i$ has no finite Intersection property.

However, if

$\bigcap_{i \in I} F_i = \emptyset \Rightarrow (\bigcap_{i \in I} F_i)^c = \emptyset^c = X$

i.e $(\bigcap_{i \in I} F_i)^c = \bigcup_{i \in I} F_i^c = X$

since F_i^c is open then

$\bigcup_{i \in I} F_i^c$ is an open cover for X and since X is Compact, then $\exists$ a finite number of the F_i^c's say $F_1^c, F_2^c, \ldots F_k^c$ which covers X.

$\Rightarrow X = \bigcup_{i=1}^{k} F_i^c$ or

$X^c = (\bigcup_{i=1}^{k} F_i^c)^c$ or $\bigcap_{i=1}^{k} F_i = \emptyset$

$\Rightarrow \bigcap F_i$ has no F.I.P

Conversely, suppose (X, τ) is not compact

$\Rightarrow \exists$ an open cover $\bigcup G_i$ of X which has no finite subcover i.e for each k,

$X \neq G_{i_1} \cup G_{i_2} \cup G_{i_3} \ldots \cup G_{i_k}$ —— (1)

If $F_i = G_i^c$ then $X^c = (\bigcup_{i \in I} G_i)^c$

$\Rightarrow \emptyset = \bigcap_{i \in I} F_i$.

However, from (i).

$$\emptyset \neq (G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_k})^c$$

$$\Rightarrow \emptyset \neq \left(\bigcup_{s=1}^{k} G_{i_s}\right)^c = \bigcap_{s=1}^{k} F_{i_s}$$

$$\Rightarrow F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \neq \emptyset$$

This shows that there is a collection of closed sets $\{F_i\}$ with F.I.P but $\bigcap_{i \in I} F_i = \emptyset$ □

DEF 7.13 : Let (X, τ) be a topological space. A subset $A \subseteq X$ is said to be locally compact if every element of A has a neighbourhood.

Example

The set $(\mathbb{R}, \mathcal{U})$ is locally compact. However $\mathbb{R}$ is not compact

DEF 7.14 : Lindelöff space

A topological space (X, τ) is said to be lindelöff space if every open cover of X can be reduced to the countable open sub-cover.

Example

The space G_2 is lindelöff. This is because for each $G \ni x \in G \; \exists \; B_i \in \mathcal{B} \ni x \in B_i \subseteq G$

Since $\mathcal{B}$ is countable then we can select the G^s to be countable as well and that gives a countable covering of X.